Handbook of Satisfiability
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IOS Press, 2008

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Chapter 6

Incomplete Algorithms

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An *incomplete* method for solving the propositional satisfiability problem (or a general constraint satisfaction problem) is one that does not provide the guarantee that it will eventually either report a satisfying assignment or declare that the given formula is unsatisfiable. In practice, most such methods are biased towards the satisfiable side: they are typically run with a pre-set resource limit, after which they either produce a valid solution or report failure; they never declare the formula to be unsatisfiable. These are the kind of algorithms we will discuss in this chapter. In complexity theory terms, such algorithms are referred to as having one-sided error. In principle, an incomplete algorithm could instead be biased towards the unsatisfiable side, always providing proofs of unsatisfiability but failing to find solutions to some satisfiable instances, or be incomplete with respect to both satisfiable and unsatisfiable instances (and thus have two-sided error).

Unlike systematic solvers often based on an exhaustive branching and backtracking search, incomplete methods are generally based on *stochastic local search*, sometimes referred to as SLS. On problems from a variety of domains, such incomplete methods for SAT can significantly outperform DPLL-based methods. Since the early 1990's, there has been a tremendous amount of research on designing, understanding, and improving local search methods for SAT.[1] There have also been attempts at hybrid approaches that explore combining ideas from DPLL methods and local search techniques [e.g. 39, 68, 84, 88]. We cannot do justice to all recent research in local search solvers for SAT, and will instead try to provide a brief overview and touch upon some interesting details. The interested reader is encouraged to further explore the area through some of the nearly a hundred publications we cite along the way.

We begin the chapter by discussing two methods that played a key role in the success of local search for satisfiability, namely `GSAT` [98] and `Walksat` [95]. We will then discuss some extensions of these ideas, in particular clause weighting

[1] For example, there is work by Anbulagan et al. [7], Cha and Iwama [14], Frank et al. [28], Gent and Walsh [31], Ginsberg and McAllester [32], Gu [37], Gu et al. [38], Hirsch and Kojevnikov [42], Hoos [44, 45], Hoos and Stützle [46], Kirkpatrick and Selman [55], Konolige [57], Li et al. [62, 63], McAllester et al. [70], Morris [78], Parkes and Walser [81], Pham et al. [83], Resende and Feo [87], Schuurmans and Southey [91], Spears [100], Thornton et al. [101], Wu and Wah [103], and others.

schemes which have made local search solvers highly competitive [14, 27, 47, 48, 78, 98, 101], and explore alternative techniques based on the discrete Lagrangian method [92, 99, 102, 103]. We will close this chapter with a discussion of the phase transition phenomenon in random k-SAT [20, 55, 76] which played a critical role in the development of incomplete methods for SAT in the 1990s, and mention a relatively new incomplete technique for SAT called Survey Propagation [73].

These ideas lie at the core of most of the local search based competitive incomplete SAT solvers out there today, and have been refined and implemented in a number of very successful prototypes including `adaptg2wsat+`, `AdaptNovelty`, `gNovelty+`, `R+AdaptNovelty+`, `SAPS`, `UBCSAT`, `UnitWalk`, and `Walksat`. Rather than going into the details of each individual solver, we hope to present the broader computer science concepts that form the backbone of these solvers and to provide the reader with enough background and references to further explore the intricacies if desired.

We note that there are other exciting incomplete solution methodologies, such as those based on translation to Integer Programming [43, 52], Evolutionary and Genetic Algorithms [22, 24, 35, 61], and Finite Learning Automata [36], that we will not discuss here. There has also been work on formally analyzing local search methods, yielding some of the best $o(2^n)$ time algorithms for SAT. For instance, the expected running time, ignoring polynomial factors, of a simple local search method with restarts after every $3n$ "flips" has been shown to be $(2 \cdot (k-1)/k)^n$ for k-SAT [89, 90], which yields a complexity of $(4/3)^n$ for 3-SAT. This result has been derandomized to yield a deterministic algorithm with complexity 1.481^n up to polynomial factors [21]. We refer the reader to Part 1, Chapter 12 of this Handbook for a detailed discussion of worst-case upper bounds for k-SAT.

For the discussion in the rest of this chapter, it will be illustrative to think of a propositional formula F with n variables and m clauses as creating a discrete manifold or *landscape* in the space $\{0,1\}^n \times \{0,1,\ldots,m\}$. The 2^n truth assignments to the variables of F correspond to the points in $\{0,1\}^n$, and the "height" of each such point, a number in $\{0,1,\ldots,m\}$, corresponds to the number of clauses of F that are violated by this truth assignment. The solutions or satisfying assignments for F are precisely the points in this landscape with height zero, and thus correspond to the global minima of this landscape, or, equivalently, the global minima of the function that maps each point in $\{0,1\}^n$ to its height. The search problem for SAT is then a search for a global minimum in this implicitly defined exponential-size landscape. Clearly, if the landscape did not have any local minima, a greedy descent would provide an effective search method. All interesting formulas, however, do have local minima—the main challenge and opportunity for the designers of local search methods.

This landscape view also leads one naturally to the problem of *maximum satisfiability* or MAX-SAT: Given a formula F, find a truth assignment that satisfies the most number of clauses possible. Solutions to the MAX-SAT problem are, again, precisely the global minima of the corresponding landscape, only that these global minima may not have height zero. Most of the incomplete methods we will discuss in this chapter, especially those based on local search, can also work as a solution approach for the MAX-SAT problem, by providing the "best found" truth assignment (i.e., one with the lowest observed height) upon termination.

Of course, while it is easy to test whether the best found truth assignment is a solution to a SAT instance, it is NP-hard to perform the same test for a MAX-SAT instance. Thus, this approach only provides a heuristic algorithm for MAX-SAT, with a two-sided error. For further details on this problem, we refer the reader to Part 2, Chapter 19 of this Handbook.

6.1. Greedy Search and Focused Random Walk

The original impetus for trying a local search method on the satisfiability problem was the successful application of such methods for finding solutions to large N-queens instances, first using a connectionist system by Adorf and Johnston [6], and then using greedy local search by Minton et al. [75]. It was originally assumed that this success simply indicated that N-queens was an easy problem, and researchers felt that such techniques would fail in practice for SAT and other more intricate problems. In particular, it was believed that local search methods would easily get stuck in local minima, with a few clauses remaining unsatisfied. Experiments with the solver `GSAT` showed, however, that certain local search strategies often do reach global minima, in many cases much faster than systematic search methods.

`GSAT` is based on a randomized local search technique [64, 80]. The basic `GSAT` procedure, introduced by Selman et al. [98] and described here as Algorithm 6.1, starts with a randomly generated truth assignment for all variables. It then greedily changes ('flips') the truth assignment of the variable that leads to the greatest decrease in the total number of unsatisfied clauses. The *neighborhood* of the current truth assignment, thus, is the set of n truth assignments each of which differs from the current one in the value of exactly one variable. Such flips are repeated until either a satisfying assignment is found or a pre-set maximum number of flips (MAX-FLIPS) is reached. This process is repeated as needed, up to a maximum of MAX-TRIES times.

Algorithm 6.1: `GSAT` (F)

```
Input      : A CNF formula F
Parameters : Integers MAX-FLIPS, MAX-TRIES
Output     : A satisfying assignment for F, or FAIL
begin
    for i ← 1 to MAX-TRIES do
        σ ← a randomly generated truth assignment for F
        for j ← 1 to MAX-FLIPS do
            if σ satisfies F then return σ                    // success
            v ← a variable flipping which results in the greatest decrease
                (possibly negative) in the number of unsatisfied clauses
            Flip v in σ
    return FAIL                          // no satisfying assignment found
end
```

Selman et al. showed that `GSAT` substantially outperformed even the best backtracking search procedures of the time on various classes of formulas, in-

cluding randomly generated formulas and SAT encodings of graph coloring instances [50]. The search of GSAT typically begins with a rapid greedy descent towards a better truth assignment (i.e., one with a lower height), followed by long sequences of "sideways" moves. Sideways moves are moves that do not increase or decrease the total number of unsatisfied clauses. In the landscape corresponding to the formula, each collection of truth assignments that are connected together by a sequence of possible sideways moves is referred to as a *plateau*. Experiments indicate that on many formulas, GSAT spends most of its time on plateaus, transitioning from one plateau to another every so often. Interestingly, Frank et al. [28] observed that in practice, almost all plateaus do have so-called "exits" that lead to another plateau with a lower number of unsatisfied clauses. Intuitively, in a very high dimensional search space such as the space of a 10,000 variable formula, it is very rare to encounter local minima, which are plateaus from where there is no local move that decreases the number of unsatisfied clauses. In practice, this means that GSAT most often does not get stuck in local minima, although it may take a substantial amount of time on each plateau before moving on to the next one. This motivates studying various modifications in order to speed up this process [96, 94]. One of the most successful strategies is to introduce noise into the search in the form of uphill moves, which forms the basis of the now well-known local search method for SAT called Walksat [95].

Walksat interleaves the greedy moves of GSAT with random walk moves of a standard Metropolis search. It further *focuses the search* by always selecting the variable to flip from an unsatisfied clause C (chosen at random). This seemingly simple idea of focusing the search turns out to be crucial for scaling such techniques to formulas beyond a few hundred variables. If there is a variable in C flipping which does not turn any currently satisfied clauses to unsatisfied, it flips this variable (a "freebie" move). Otherwise, with a certain probability, it flips a random literal of C (a "random walk" move), and with the remaining probability, it flips a variable in C that minimizes the *break-count*, i.e., the number of currently satisfied clauses that become unsatisfied (a "greedy" move). Walksat is presented in detail as Algorithm 6.2. One of its parameters, in addition to the maximum number of tries and flips, is the *noise* $p \in [0, 1]$, which controls how often are non-greedy moves considered during the stochastic search. It has been found empirically that for various instances from a single domain, a single value of p is optimal. For random 3-SAT formulas, the optimal noise is seen to be 0.57, and at this setting, Walksat is empirically observed to scale linearly for clause-to-variable ratios α up to (and even slightly beyond) 4.2 [93], though not all the way up to the conjectured satisfiability threshold of nearly 4.26.[2]

The focusing strategy of Walksat based on selecting variables solely from unsatisfied clauses was inspired by the $O(n^2)$ randomized algorithm for 2-SAT by Papadimitriou [79]. It can be shown that for any satisfiable formula and starting from any truth assignment, there exists a sequence of flips using only variables from unsatisfied clauses such that one obtains a satisfying assignment.

Remark 6.1.1. We take a small detour to explain the elegant and insightful $O(n^2)$ time randomized local search algorithm for 2-SAT by Papadimitriou [79].

[2] Aurell et al. [8] had observed earlier that Walksat scales linearly for random 3-SAT at least till clause-to-variable ratio 4.15.

Algorithm 6.2: `Walksat` (F)

```
Input      : A CNF formula F
Parameters : Integers MAX-FLIPS, MAX-TRIES; noise parameter p ∈ [0, 1]
Output     : A satisfying assignment for F, or FAIL
begin
    for i ← 1 to MAX-TRIES do
        σ ← a randomly generated truth assignment for F
        for j ← 1 to MAX-FLIPS do
            if σ satisfies F then return σ                          // success
            C ← an unsatisfied clause of F chosen at random
            if ∃ variable x ∈ C with break-count = 0 then
                v ← x                                               // freebie move
            else
                With probability p:                                 // random walk move
                    v ← a variable in C chosen at random
                With probability 1 − p:                             // greedy move
                    v ← a variable in C with the smallest break-count
            Flip v in σ
    return FAIL                                                     // no satisfying assignment found
end
```

The algorithm itself is very simple: while the current truth assignment does not satisfy all clauses, select an unsatisfied clause arbitrarily, select one of its variables uniformly at random, and flip this variable. Why does this take $O(n^2)$ flips in expectation before finding a solution? Assume without loss of generality that the all-zeros string (i.e., all variables set to FALSE) is a satisfying assignment for the formula at hand. Let $\bar{\sigma}$ denote this particular solution. Consider the Hamming distance $d(\sigma, \bar{\sigma})$ between the current truth assignment, σ, and the (unknown) solution $\bar{\sigma}$. Note that $d(\sigma, \bar{\sigma})$ equals the number of TRUE variables in σ. We claim that in each step of the algorithm, we reduce $d(\sigma, \bar{\sigma})$ by 1 with probability at least a half, and increase it by one with probability less than a half. To see this, note that since $\bar{\sigma}$ is a solution, all clauses of the formula have at least one negative literal so that any unsatisfied clause selected by the algorithm must involve at least one variable that is currently set to TRUE, and selecting this variable to flip will result in decreasing $d(\sigma, \bar{\sigma})$ by 1. Given this claim, the algorithm is equivalent to a one-dimensional Markov chain of length $n+1$ with the target node—the all-zeros string—on one extreme and the all-ones string at the other extreme, and that at every point we walk towards the target node with probability at least a half. It follows from standard Markov chain hitting time analysis that we will hit $\bar{\sigma}$ after $O(n^2)$ steps. (The algorithm could, of course, hit another solution before hitting $\bar{\sigma}$, which will reduce the runtime even further.)

When one compares the biased random walk strategy of `Walksat` on hard random 3-CNF formulas against basic `GSAT`, the simulated annealing process of Kirkpatrick et al. [54], and a pure random walk strategy, the biased random walk process significantly outperforms the other methods [94]. In the years follow-

ing the development of `Walksat`, many similar methods have been shown to be highly effective on not only random formulas but on several classes of structured instances, such as encodings of circuit design problems, Steiner tree problems, problems in finite algebra, and AI planning [cf. 46].

6.2. Extensions of the Basic Local Search Method

Various extensions of the basic process discussed above have been explored, such as dynamic noise adaptation as in the solver `adapt-novelty` [45], incorporating unit clause elimination as in the solver `UnitWalk` [42], incorporating resolution-based reasoning [7], and exploiting problem structure for increased efficiency [83]. Recently, it was shown that the performance of stochastic solvers on many structured problems can be further enhanced by using new SAT encodings that are designed to be effective for local search [85].

While adding random walk moves as discussed above turned out to be a successful method of guiding the search away from local basins of attraction and toward other parts of the search space, a different line of research considered techniques that relied on the idea of *clause re-weighting* as an extension of basic greedy search [14, 27, 47, 48, 78, 98, 101]. Here one assigns a positive weight to each clause and attempts to minimize the sum of the weights of the unsatisfied clauses. The clause weights are dynamically modified as the search progresses, increasing the weight of the clauses that are currently unsatisfied. (In some implementations, increasing the weight of a clause is done by simply adding identical copies of the clause.) In this way, if one waits sufficiently long, any unsatisfied clause gathers enough weight so as to sway the truth assignment in its favor. This is thought of as "flooding" the current local minimum by re-weighting unsatisfied clauses to create a new descent direction for local search. Variants of this approach differ in the re-weighting strategy used, e.g., how often and by how much the weights of unsatisfied clauses are increased, and how are all weights periodically decreased in order to prevent certain weights from becoming dis-proportionately high. The work on DLM or Discrete Lagrangian Method grounded these techniques in a solid theoretical framework, whose details we defer to Section 6.3. The SDF or "smoothed descent and flood" system of Schuurmans and Southey [91] achieved significantly improved results by using multiplicative (rather than additive) re-weighting, by making local moves based on how strongly are the clauses currently satisfied in terms of the number of satisfied literals (rather than simply how many are satisfied), and by periodically shrinking all weights towards their common mean. Overall, two of the most well-known clause-reweighting schemes that have been proposed are SAPS (*scaling and probabilistic smoothing*, along with its *reactive* variant RSAPS) [47] and PAWS (*pure additive weighting scheme*) [101].

Other approaches for improving the performance of `GSAT` were also explored in the early years. These include `TSAT` by Mazure et al. [69], who maintain a tabu list in order to prevent `GSAT` from repeating earlier moves, and `HSAT` by Gent and Walsh [31], who consider breaking ties in favor of least recently flipped variables. These strategies provide improvement, but to a lesser extent than the basic random walk component added by `Walksat`.

In an attempt towards better understanding the pros and cons of many of these techniques, Schuurmans and Southey [91] proposed three simple, intuitive measures of the effectiveness of local search: *depth, mobility,* and *coverage*. (A) Depth measures how many clauses remain unsatisfied as the search proceeds. Typically, good local search strategies quickly descend to low depth and stay there for a long time; this corresponds to spending as much time as possible near the bottom of the search landscape. (B) Mobility measures how rapidly the process moves to new regions in the search space (while simultaneously trying to stay deep in the objective). Clearly, the larger the mobility, the better the chance that a local search strategy has of achieving success. (C) Coverage measures how systematically the process explores the entire space, in terms of the largest "gap", i.e., the maximum Hamming distance between any unexplored assignment and the nearest evaluated assignment. Schuurmans and Southey [91] hypothesized that, in general, successful local search procedures work well not because they possess any special ability to predict whether a local basin of attraction contains a solution or not—rather they simply descend to promising regions and explore near the bottom of the objective as rapidly, broadly, and systematically as possible, until they stumble across a solution.

6.3. Discrete Lagrangian Methods

Shang and Wah [99] introduced a local search system for SAT based on the theory of Lagrange multipliers. They extended the standard Lagrange method, traditionally used for continuous optimization, to the discrete case of propositional satisfiability, in a system called DLM (Discrete Lagrangian Method). Although the final algorithm that comes out of this process can be viewed as a clause weighted version of local search as discussed in Section 6.2, this approach provided a theoretical foundation for many design choices that had appeared somewhat ad-hoc in the past. The change in the weights of clauses that are unsatisfied translates in this system to a change is the corresponding Lagrange multipliers, as one searches for a (local) optimum of the associated Lagrange function.

The basic idea behind DLM is the following. Consider an n-variable CNF formula F with clauses $C_1, C_2, \ldots, C_m$. For $x \in \{0,1\}^n$ and $i \in \{1, 2, \ldots, m\}$, let $U_i(x)$ be a function that is 0 if C_i is satisfied by x, and 1 otherwise. Then the SAT problem for F can be written as the following optimization problem over $x \in \{0,1\}^n$:

$$\begin{aligned} \text{minimize} \quad & N(x) = \sum_{i=1}^{m} U_i(x) \\ \text{subject to} \quad & U_i(x) = 0 \qquad\qquad \forall i \in \{1, 2, \ldots, m\} \end{aligned} \tag{6.1}$$

Notice that $N(x) \geq 0$ and equals 0 if and only if all clauses of F are satisfied. Thus, the objective function $N(x)$ is minimized if and only if x is a satisfying assignment for F. This formulation, somewhat strangely, also has each clause as an explicit constraint $U_i(x)$, so that any feasible solution is automatically also locally as well as globally optimal. This redundancy, however, is argued to be the strength of the system: the dynamic shift in emphasis between the objective and

the constraints, depending on the relative values of the Lagrange multipliers, is the key feature of Lagrangian methods.

The theory of discrete Lagrange multipliers provides a recipe for converting the above constrained optimization problem into an unconstrained optimization problem, by introducing a Lagrange multiplier for each of the constraints and adding the constraints, appropriately multiplied, to the objective function. The resulting discrete Lagrangian function, which provides the new objective function to be optimized, is similar to what one would obtain in the traditional continuous optimization case:

$$L_d(x, \lambda) = N(x) + \sum_{i=1}^{m} \lambda_i U_i(x) \tag{6.2}$$

where $x \in \{0,1\}^n$ are points in the variable space and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$ is a vector of Lagrange multipliers, one for each constraint in (i.e., clause of) F. A point $(x^*, \lambda^*) \in \{0,1\}^n \times \mathbb{R}^m$ is called a *saddle point* of the Lagrange function $L_d(x, \lambda)$ if it is a local minimum w.r.t. x^* and a local maximum w.r.t. λ^*. Formally, (x^*, λ^*) is a saddle point for L_d if

$$L_d(x^*, \lambda) \;\leq\; L_d(x^*, \lambda^*) \;\leq\; L_d(x, \lambda^*)$$

for all λ sufficiently close to λ^* and for all x that differ from x^* only in one dimension. It can be proven that x^* is a local minimum solution to the discrete constrained optimization formulation of SAT (6.1) if there exists a λ^* such that (x^*, λ^*) is a saddle point of the associated Lagrangian function $L_d(x, \lambda)$. Therefore, local search methods based on the Lagrangian system look for saddle points in the extended space of variables and Lagrange multipliers. By doing descents in the original variable space and ascents in the Lagrange-multiplier space, a saddle point equilibrium is reached when (locally) optimal solutions are found.

For SAT, this is done through a *difference gradient* $\Delta_x L_d(x, \lambda)$, defined to be a vector in $\{-1, 0, 1\}^n$ with the properties that it has at most one non-zero entry and $y = x \oplus \Delta_x L_d(x, \lambda)$ minimizes $L_d(y, \lambda)$ over all neighboring points y of x, including x itself. Here $\oplus$ denotes vector addition; $x \oplus z = (x_1 + z_1, \ldots, x_n + z_n)$. The neighbors are "user-defined" and are usually taken to be all points that differ from x only in one dimension (i.e., are one variable flip away). Intuitively, the difference gradient $\Delta_x L_d(x, \lambda)$ "points in the direction" of the neighboring value in the variable space that minimizes the Lagrangian function for the current λ.

This yields an algorithmic approach for minimizing the discrete Lagrangian function $L_d(x, \lambda)$ associated with SAT (and hence minimizing the objective function $N(x)$ in the discrete constrained optimization formulation of SAT (6.1)). The algorithm proceeds iteratively in stages, updating $x \in \{0,1\}^n$ and $\lambda \in \mathbb{R}^m$ in each stage using the difference gradient and the current status of each constraint in terms of whether or not it is satisfied, until a fixed point is found. Let $x(k)$ and $\lambda(k)$ denote the values of x and λ after the k^{th} iteration of the algorithm. Then the updates are as follows:

$$\begin{aligned} x(k+1) &= x(k) \oplus \Delta_x L_d(x(k), \lambda(k)) \\ \lambda(k+1) &= \lambda(k) + c\, U(x(k)) \end{aligned}$$

where $c \in \mathbb{R}^+$ is a parameter that controls how fast the Lagrange multipliers are increased over iterations and U denotes the vector of the m constraint functions U_i defined earlier. The difference gradient $\Delta_x L_d$ determines which variable, if any, to flip in order to lower $L_d(x, \lambda)$ for the current λ. When a fixed point for these iterations is reached, i.e., when $x(k+1) = x(k)$ and $\lambda(k+1) = \lambda(k)$, it must be that all constraints U_i are satisfied. To see this, observe that if the i^{th} clause is unsatisfied after the k^{th} iteration, then $U_i(x(k)) = 1$, which implies $\lambda_i(k+1) = \lambda_i(k) + c$; thus, λ_i will keep increasing until U_i is satisfied. This provides *dynamic clause re-weighting* in this system, placing more and more emphasis on clauses that are unsatisfied until they become satisfied. Note that changing the Lagrange multipliers λ_i in turn affects x by changing the direction in which the difference gradient $\Delta_x L_d(x, \lambda)$ points, eventually leading to a point at which all constraints are satisfied. This is the essence of the DLM system for SAT.

The basic implementation of DLM [99] uses a simple controlled update rule for λ: increase λ_i by 1 for all unsatisfied clauses C_i after a pre-set number θ_1 of *up-hill* or *flat* moves (i.e., changes in x that do not decrease $L_d(x, \lambda)$) have been performed. In order to avoid letting some Lagrange multipliers become disproportionately large during the search, all λ_i's are periodically decreased after a pre-set number θ_2 of increases in λ have been performed. Finally, the implementation uses *tabu lists* to store recently flipped variables, so as to avoid flipping them repeatedly.

Wu and Wah [102] observed that in many difficult to solve instances, such as from the `parity` and `hanoi` domains, basic DLM frequently gets stuck in *traps*, i.e., pairs (x, λ) such that there are one or more unsatisfied clauses but the associated L_d increases by changing x in any one dimension. They found that on these instances, some clauses are significantly more likely than others to be amongst the unsatisfied clauses in such a trap. (Note that this is different from counting how often is a clause unsatisfied; here we only consider the clause status inside a trap situation, ignoring how the search arrived at the trap.) Keeping track of such clauses and periodically performing a special increase in their associated Lagrange multipliers, guides the search away from traps and results in better performance.

[103] later generalized this strategy by recording not only information about traps but a history of all recently visited points in the variable space. Instead of performing a special increase periodically, this history information is now added directly as a *distance penalty* term to the Lagrangian function L_d. The penalty is larger for points that are in Hamming distance closer to the current value of x, thereby guiding the search away from recently visited points (in particular, points inside a trap that would be visited repeatedly).

6.4. The Phase Transition Phenomenon in Random *k*-SAT

One of the key motivations in the early 1990's for studying incomplete, stochastic methods for solving SAT instances was the observation that DPLL-based systematic solvers perform quite poorly on certain randomly generated formulas. For completeness, we provide a brief overview of these issues here; for a detailed discussion, refer to Part 1, Chapter 8 of this Handbook.

Consider a random k-CNF formula F on n variables generated by independently creating m clauses as follows: for each clause, select k distinct variables uniformly at random out of the n variables and negate each selected variable independently with probability 1/2. When F is chosen from this distribution, Mitchell, Selman, and Levesque [76] observed that the median hardness of the instances is very nicely characterized by a single parameter: the *clause-to-variable ratio*, m/n, typically denoted by α. They observed that instance hardness peaks in a critically constrained region determined by α alone. The top pane of Figure 6.1 depicts the now well-known "easy-hard-easy" pattern of SAT and other combinatorial problems, as the key parameter (in this case α) is varied. For random 3-SAT, this region has been experimentally shown to be around $\alpha \approx 4.26$ (for early results see Crawford and Auton [20], Kirkpatrick and Selman [55]; new findings by Mertens et al. [71]), and has provided challenging benchmarks as a test-bed for SAT solvers. Cheeseman et al. [15] observed a similar easy-hard-easy pattern in the random graph coloring problem. For random formulas, interestingly, a slight natural variant of the above "fixed-clause-length" model, called the variable-clause-length model, does *not* have a clear set of parameters that leads to a hard set of instances [26, 33, 86]. This apparent difficulty in generating computationally hard instances for SAT solvers provided the impetus for much of the early work on local search methods for SAT. We refer the reader to Cook and Mitchell [19] for a detailed survey.

The critically constrained region marks a stark transition not only in the computational hardness of random SAT instances but also in their satisfiability itself. The bottom pane of Figure 6.1 shows the fraction of random formulas that are unsatisfiable, as a function of α. We see that nearly all formulas with α below the critical region (the under-constrained instances) are satisfiable. As α approaches and passes the critical region, there is a sudden change and nearly all formulas in this over-constrained region are unsatisfiable. Further, as n grows, this phase transition phenomenon becomes sharper and sharper, and coincides with the region in which the computational hardness peaks. The relative hardness of the instances in the unsatisfiable region to the right of the phase transition is consistent with the formal result of Chvátal and Szemerédi [17] who, building upon the work of Haken [41], proved that large unsatisfiable random k-CNF formulas almost surely require exponential size resolution refutations, and thus exponential length runs of any DPLL-based algorithm proving unsatisfiability. This formal result was subsequently refined and strengthened by others [cf. 9, 10, 18].

Relating the phase transition phenomenon for 3-SAT to statistical physics, Kirkpatrick and Selman [55] showed that the threshold has characteristics typical of phase transitions in the statistical mechanics of disordered materials (see also Monasson et al. [77] and Part 2, Chapter 18 of this Handbook). Physicists have studied phase transition phenomena in great detail because of the many interesting changes in a system's macroscopic behavior that occur at phase boundaries. One useful tool for the analysis of phase transition phenomena is called *finite-size scaling* analysis [97]. This approach is based on rescaling the horizontal axis by a factor that is a function of n. The function is such that the horizontal axis is stretched out for larger n. In effect, rescaling "slows down" the phase-transition

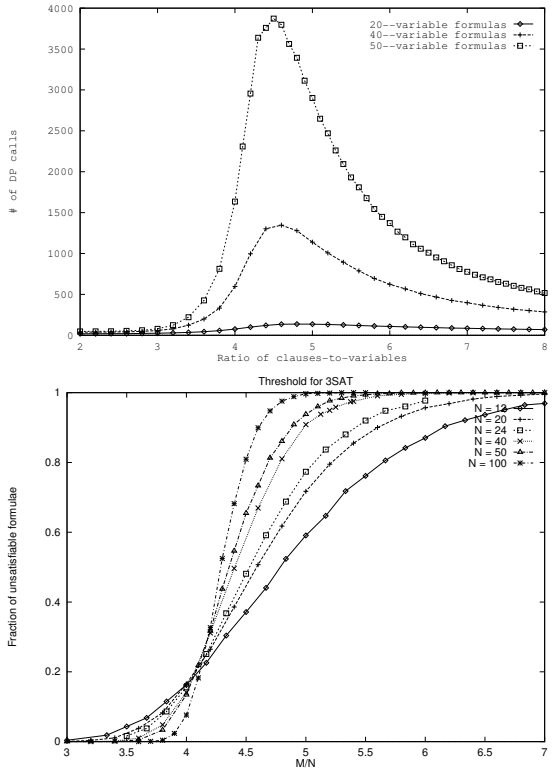

Figure 6.1. The phase transition phenomenon in random 3-SAT. Top: Computational hardness peaks at $\alpha \approx 4.26$. Bottom: Formulas change from being mostly satisfiable to mostly unsatisfiable. The transitions sharpen as the number of variables grows.

for higher values of n, and thus gives us a better look inside the transition. From the resulting universal curve, applying the scaling function backwards, the actual transition curve for each value of n can be obtained. In principle, this approach also localizes the 50%-satisfiable-point for any value of n, which allows one to generate very hard random 3-SAT instances.

Interestingly, it is still not formally known whether there even exists a critical constant α_c such that as n grows, almost all 3-SAT formulas with $\alpha < \alpha_c$ are satisfiable and almost all 3-SAT formulas with $\alpha > \alpha_c$ are unsatisfiable. In this respect, Friedgut [29] provided the first positive result, showing that there exists a *function* $\alpha_c(n)$ depending on n such that the above threshold property holds. (It is quite likely that the threshold in fact does not depend on n, and is a fixed constant.) In a series of papers, researchers have narrowed down the gap between upper bounds on the threshold for 3-SAT [e.g. 13, 23, 26, 49, 56], the best so far being 4.51 [23], and lower bounds [e.g. 1, 5, 13, 25, 30, 40, 53], the best so far being 3.52 [40, 53]. On the other hand, for random 2-SAT, we do have a full rigorous understanding of the phase transition, which occurs at the clause-to-

variable ratio of 1 [11, 16]. Also, for general k, the threshold for random k-SAT is known to be in the range $2^k \ln 2 - \mathrm{O}(k)$ [2, 3, 34].

6.5. A New Technique for Random k-SAT: Survey Propagation

We end this chapter with a brief discussion of Survey Propagation (SP), an exciting new incomplete algorithm for solving hard combinatorial problems. The reader is referred to Part 2, Chapter 18 of this Handbook for a detailed treatment of work in this direction. Survey propagation was discovered in 2002 by Mézard, Parisi, and Zecchina [73], and is so far the only known method successful at solving random 3-SAT instances with one million variables and beyond in near-linear time in the most critically constrained region.[3]

The SP method is quite radical in that it tries to approximate, using an iterative process of local "message" updates, certain marginal probabilities related to the set of satisfying assignments. It then assigns values to variables with the most extreme probabilities, simplifies the formula, and repeats the process. This strategy is referred to as SP-inspired decimation. In effect, the algorithm behaves like the usual DPLL-based methods, which also assign variable values incrementally in an attempt to find a satisfying assignment. However, quite surprisingly, SP almost never has to backtrack. In other words, the "heuristic guidance" from SP is almost always correct. Note that, interestingly, computing marginals on satisfying assignments is strongly believed to be much harder than finding a single satisfying assignment (#P-complete vs. NP-complete). Nonetheless, SP is able to efficiently approximate certain marginals on random SAT instances and uses this information to successfully find a satisfying assignment.

SP was derived from rather complex statistical physics methods, specifically, the so-called *cavity method* developed for the study of spin glasses. The origin of SP in statistical physics and its remarkable and unparalleled success on extremely challenging random 3-SAT instances has sparked a lot of interest in the computer science research community, and has led to a series of papers in the last five years exploring various aspects of SP [e.g. 4, 8, 12, 58–60, 65, 67, 66, 72, 74, 104]. Many of these aspects still remain somewhat mysterious, making SP an active and promising research area for statistical physicists, theoretical computer scientists, and artificial intelligence practitioners alike.

While the method is still far from well-understood, close connections to belief propagation (BP) methods [82] more familiar to computer scientists have been subsequently discovered. In particular, it was shown by Braunstein and Zecchina [12] (later extended by Maneva, Mossel, and Wainwright [66]) that SP equations are equivalent to BP equations for obtaining marginals over a special class of combinatorial objects called covers. In this respect, SP is the first successful example of the use of a probabilistic reasoning technique to solve a purely combinatorial search problem. The recent work of Kroc et al. [58] empirically established that SP, despite the very loopy nature of random formulas which violate the standard tree-structure assumptions underlying the BP algorithm, is remarkably good at

[3] As mentioned earlier, it has been recently shown that by finely tuning the noise and temperature parameters, `Walksat` can also be made to scale well on hard random 3-SAT instances with clause-to-variable ratios α slightly exceeding 4.2 [93].

computing marginals over these covers objects on large random 3-SAT instances. Kroc et al. [59] also demonstrated that information obtained from BP-style algorithms can be effectively used to enhance the performance of algorithms for the model counting problem, a generalization of the SAT problem where one is interested in counting the number of satisfying assignments.

Unfortunately, the success of SP is currently limited to random SAT instances. It is an exciting research challenge to further understand SP and apply it successfully to more structured, real-world problem instances.

6.6. Conclusion

Incomplete algorithms for satisfiability testing provide a complementary approach to complete methods, using an essentially disjoint set of techniques and being often well-suited to problem domains in which complete methods do not scale well. While a mixture of greedy descent and random walk provide the basis for most local search SAT solvers in use today, much work has gone into finding the right balance and in developing techniques to focus the search and efficiently bring it out of local minima and traps. Formalisms like the discrete Lagrangian method and ideas like clause weighting or flooding have played a crucial role in pushing the understanding and scalability of local search methods for SAT. An important role has also been played by the random k-SAT problem, particularly in providing hard benchmarks and a connection to the statistical physics community, leading to the survey propagation algorithm. Can we bring together ideas and techniques from systematic solvers and incomplete solvers to create a solver that has the best of both worlds? While some progress has been made in this direction, much remains to be done.

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Index